Every planar graph without adjacent short cycles is 3-colorable

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Abstract

Two cycles are *adjacent* if they have an edge in common. Suppose that G is a planar graph, for any two adjacent cycles C_1 and C_2, we have $|C_1| + |C_2| \geq 11$, in particular, when $|C_1| = 5$, $|C_2| \geq 7$. We show that the graph G is 3-colorable.

1 Introduction

In 1852, Francis Guthrie proposed the Four Color Problem. In 1976, K. Appel and W. Haken proved the Four Color Theorem:

Theorem 1.1. Every planar graph is 4-colorable.

In 1976, Garey et al. [9] proved the problem of deciding whether a planar graph is 3-colorable is NP-complete. In 1959, Grötzsch [10] showed that every planar graph without 3-cycles is 3-colorable. In 1976, Steinberg conjectured the following:

Conjecture 1 (Steinberg's Conjecture)**.** Every planar graph without 4- and 5-cycles is 3-colorable.

This conjecture remains open. In 1991, Erdös suggested the following relaxation of Steinberg's Conjecture by asking whether there exists an integer k such that the absence of cycles of lengths from 4 to k in a planar graph guarantees its 3-colorability.

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Abbott and Zhou [1] proved such an integer k exists and $k \leq 11$. The bound on integer k was later improved to 10 by Borodin [3], to 9 by Borodin [2] and, independently, by Sanders and Zhao [12], to 8 by Salavatipour [11], to 7 by Borodin et al. [6].

Towards Steinberg's Conjecture, one direction is to show that planar graph without adjacent short cycles is 3-colorable, for instance, the following result is such an attempt:

Theorem 1.2. Every planar graph without 5- and 7-cycles and without adjacent triangles is 3-colorable.

Note that the first attempt to prove this theorem was made by Xu [13], but his proof was not correct. Borodin et al. gave a new proof of Theorem 1.2, see [5].

Recent progress are presented the service of theorems below.

Theorem 1.3 (Borodin et al. [4])**.** Every planar graph without triangles adjacent to cycles of length from 3 to 9 is 3-colorable.

Theorem 1.4 (Borodin et al. [7])**.** Every planar graph in which no i-cycle is adjacent to a j-cycle whenever $3 \leq i \leq j \leq 7$ is 3-colorable.

Conjecture 2 (Strong Bordeaux Conjecture [8])**.** Every planar graph without 5-cycles and without adjacent triangles is 3-colorable.

Conjecture 3 (Novosibirsk 3-Color Conjecture, [4])**.** Every planar graph without 3-cycles adjacent to 3-cycles or 5-cycles is 3-colorable.

2 Preliminaries

In this paper, the graphs considered may contain multiple edges, but no loops. The *neighborhood* of a vertex $v \in V(G)$, denoted by $N_G(v)$, is the set of all the vertices adjacent to v, i.e., $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$. The *degree* of a vertex v in G, denoted by $\deg_G(v)$, is the number of its neighbors in G, i.e., $\deg_G(v) = |N_G(v)|$. A vertex of degree k is also referred as a *k-vertex*. Two cycles are *adjacent* if they have an edge in common.

For a plane graph, the edges and vertices divide the plane into a number of *faces*. The unbounded face is called the *outer face*, and the others are called *inner faces*. The boundary of the outer face of G is called the *outer boundary* of G and denoted by $C_0(G)$. If $C_0(G)$ is a cycle, then $C_0(G)$ is called the *outer cycle* of G. We call a vertex v of G an *outer vertex* of G if v is on $C_0(G)$; otherwise v is an *inner vertex* of G. Similarly we define an outer edge and an inner edge of G. The *degree of a face F* of G is the number of edges in its boundary, counting those edges twice for which F lies on both sides. A k-face is a face of degree k. A face is said to be *incident* with vertices and edges in its boundary, and two faces are *adjacent* if their boundaries have an edge in common. A vertex is *bad* if it is an

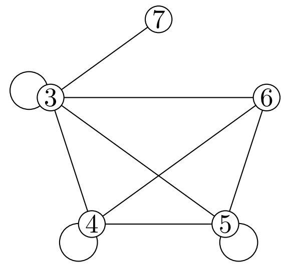


Fig. 1: The nonadjacency graph $G_{\mathcal{A}}$.

inner 3-vertex and is incident with a triangle. Let C be a cycle of a plane graph G. The cycle C divides the plane into two regions, the unbounded region is denoted by $\mathrm{ext}(C)$, and the other region is denoted by $\mathrm{int}(C)$. If both $\mathrm{int}(C)$ and $\mathrm{ext}(C)$ contain at least one vertex, then we say that the cycle C is a *separating cycle* of G. Let u and v be two vertices of a cycle C in G, the segment of C clockwisely from u to v is denoted by $C[u, v]$, and $C(u, v) = C[u, v] - \{u, v\}$.

A *nonadjacency* graph is one whose vertices are labeled by integers greater than two and each integer appears at most once. Given a graph $G_{\mathcal{A}}$ of nonadjacency, we say that a graph G belongs to $G_{\mathcal{A}}$ or G has the nonadjacency property $\mathcal{A}$ if no two cycles of lengths i and j are adjacent in G when the vertices labeled with i and j are adjacent in $G_{\mathcal{A}}$.

Let $\mathcal{G}_{(A)}$ be the class of graphs belongs to the nonadjacency graph depicted in Fig. 1. In this paper, we prove the following.

Theorem 2.1. Every planar graph in $\mathcal{G}_{(A)}$ is 3-colorable.

3 Proof of the main result

In attempt to prove Theorem 2.1, we prove a strong color extension lemma.

Lemma 1. Suppose that G is a plane graph in $\mathcal{G}_{(A)}$, and f_0 is the outer face of G with degree at most 11, then every proper 3-coloring of $G[V(f_0)]$ can be extend to a proper 3-coloring of G.

Proof. By way of contradiction, we assume that the result is not true. Let G be a counterexample to the Lemma with the following condition: $|V(G)| + |E(G)|$ is minimum among all the counterexamples. Let C_0 be the boundary of the outer face f_0. Then there exists a proper 3-coloring of $G[V(f_0)]$ which cannot be extended to a proper 3-coloring of G. Moreover, the minimum counterexample G has the following properties.

(1) The graph G is simple, i.e., it has no loops and no multiple edges.

(2) $\mathrm{int}(C_0)$ contains at least one vertex.

(3) For every vertex v in $\mathrm{int}(C_0)$, the degree of v in G is at least three.

(4) The graph G is 2-connected, and thus the boundary of each face is a cycle.

From now on, for any integer $i \geq 4$, i^- denotes every positive integer ranges from 3 to i and i^+ denotes all the positive integer greater than i.

(5) The graph G has no separating cycles of length at most eleven. So every 11^--cycle is a facial cycle.

(6) The outer cycle C_0 has no chords. For any inner face f of G, at least one vertex of the boundary of f is not on C_0.

Proof. Let xy be a chord of the outer cycle C_0. By the minimality of G, the 3-coloring of $G[V(f_0)]$ can be extend to a proper 3-coloring of $G - xy$. Obviously, it is also a proper 3-coloring of G. □

(7) If C is a cycle of length at most 11, then every vertex in $\mathrm{int}(C)$ has at most two neighbors on C.

Proof. If v has three neighbors on the cycle C, then the vertex v and its three incident edges partition the cycle into three cycles. According to the lengths of the smallest cycle, there are several cases. If the smallest one is of length three, the other two are of length at least eight as $G \in \mathcal{G}_{(A)}$, then $|C| \geq 3 + 8 + 8 - 6 = 13$, a contradiction. If the smallest one is of length four, the other two are of length at least seven, then $|C| \geq 4 + 7 + 7 - 6 = 12$, a contradiction. If the smallest one is of length five, the other two are of length at least seven, then $|C| \geq 5 + 7 + 7 - 6 = 13$, a contradiction. If the smallest one is of length no less than six, then $|C| \geq 6 + 6 + 6 - 6 = 12$, a contradiction. □

(8) If C is a cycle of length at most 11, then every vertex in $\mathrm{int}(C)$ has at most one neighbor on C, except when $|C| = 11$ and the two neighbors on C are consecutive.

Proof. Suppose that there exists a vertex v in $\mathrm{int}(C)$ such that it has two neighbors v_1 and v_2 on the cycle C. By (7), the vertices v_1 and v_2 are the only two neighbors on C; and the path v_1vv_2 split the cycle C into two cycles $C_1 = vC[v_1, v_2]v$ and $C_2 = vC[v_2, v_1]v$. Clearly, the vertex v is in $\mathrm{int}(C_0)$, so $\deg_G(v) \geq 3$ and v has at least one neighbor in $\mathrm{int}(C)$. Then at least one of C_i $(i = 1, 2)$, say C_1, is a separating cycle. It follows from (5) that $|C_1| \geq 12$. Hence $|C_2| = 3$ and $|C| = 11$. □

(9) Let f be a face with boundary $\partial(f) = v_0v_1v_2 \dots v_lv_0$. Assume that $v_1, v_2, \dots, v_k$ (where $k \geq 3$) are inner vertices consecutively on the boundary, and they are all of degree three. If the edge v_1v_2 is in a triangle $v_1w_1v_2v_1$ and the other neighbor of v_3 is w_2, then the distance

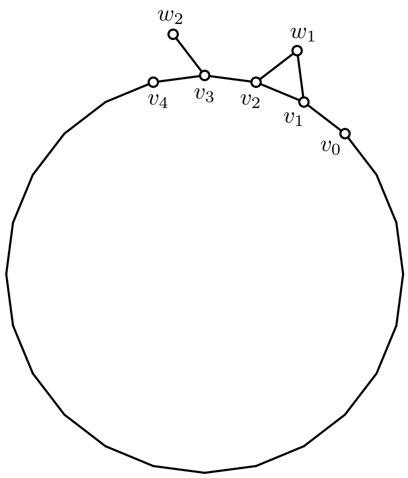


Fig. 2: A local structure in (9)

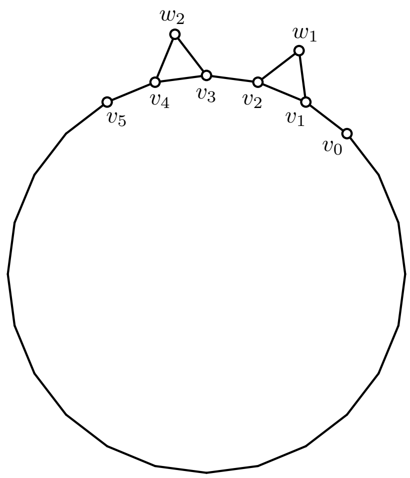


Fig. 3: A tetrad

between v_0 and w_2 in the graph $G - \{v_1, v_2, \ldots, v_k\}$ is at most seven, and $k = 3$, see Fig. 2. Moreover, vertices w_2, v_3, v_4 are consecutively on the boundary of a 5^--face.

Proof. Let G^* be the graph obtained from G by deleting vertices $v_1, v_2, \ldots, v_k$ and identifying vertex v_0 with vertex w_2.

In the following proof, we will frequently use the fact that $G \in \mathcal{G}_{(A)}$ and the triangle $v_1w_1v_2v_1$ is not adjacent to any 7^--cycle.

First, we show that the distance between v_0 and w_2 in the graph $G - \{v_1, v_2, \ldots, v_k\}$ is at most seven. If the distance is greater than seven, then the identification does not create new cycles of length at most seven, and hence $G^* \in \mathcal{G}_{(A)}$. Moreover, the cycle C_0 is also the outer cycle of G^*, and the identification does not create chords of C_0. By the minimality of G, the precoloring of C_0 can be extend to a proper 3-coloring of G^*, and then a proper 3-coloring of G, a contradiction. So we may assume that the distance between v_0 and w_2 in the graph $G - \{v_1, v_2, \ldots, v_k\}$ is at most seven.

Let P be a shortest path between v_0 and w_2 in the graph $G - \{v_1, v_2, \ldots, v_k\}$. It is easy to see that w_1 is not on the path P. If v_4 is not on the path P, then the cycle $Pw_2v_3v_2v_1v_0$ is a cycle of length at most eleven separating w_1 from v_4. Therefore, the vertex v_4 is on the path P, and hence $k = 3$. The cycle $P[v_0, v_4]v_3v_2v_1v_0$ have a common edge with the triangle $v_1w_1v_2v_1$, so $|P[v_0, v_4]| \geq 4$, and then $|P[w_2, v_4]| \leq 3$. Therefore, $P[w_2, v_4]v_3w_2$ is a cycle of length at most five, by (5), it bounds an inner face of degree at most five. □

By (9), if $v_4v_3w_2$ is also a 3-cycle, then v_4 is on C_0 or has degree at least four.

A *tetrad* is a local structure having four bad vertices v_1, v_2, v_3 and v_4 consecutively on the boundary of a face (the degree of the face is at least six) with the edge v_1v_2 in a triangle and the edge v_3v_4 in a triangle (see Fig. 3).

(10) The graph G contains no tetrad.

It follows from (9) and (10) that:

(11) A face doesn't have five bad vertices consecutively on the boundary.

(12) The graph G has no inner 4-faces.

Proof. Suppose that f is an inner 4-face, and the boundary of f is a 4-cycle $\partial(f) = v_1v_2v_3v_4v_1$ (the v_i's appearing counterclockwise on f). Let G^* be the graph obtained from G by identifying the vertices v_1 with v_3.

First, we show that the identification does not damage the outer cycle C_0. Otherwise, both v_1 and v_3 are on the outer cycle C_0, by (6), one of $\{v_2, v_4\}$, say v_2, is not on C_0. Then by (8), v_2 has two neighbors consecutive on C_0, that is, v_1 and v_3 are adjacent in G, contradicting the fact that 4-cycles are chordless. Therefore, C_0 is also the outer cycle of G^*.

Assume that the identification create a new chord of C_0. Without loss of generality, assume that v_3 is on C_0, but v_1 is not on C_0 and v_1 has a neighbor on C_0, say v. Since the edge v_1v_2 is in the 4-cycle $v_1v_2v_3v_4v_1$, then it can not be contained in a 3-cycle, by (8), the vertex v_2 is in $\mathrm{int}(C_0)$. Similarly, the vertex v_4 is in $\mathrm{int}(C_0)$. The cycle $C_0[v, v_3]v_4v_1v$ is a separating cycle of G, then $|C_0[v, v_3]| \geq 9$. Similarly, the cycle $C_0[v_3, v]v_1v_2v_3$ is a separating cycle of G, and $|C_0[v_3, v]| \geq 9$, then $|C_0| \geq 9 + 9 = 18$, a contradiction.

Let C^* be an arbitrary new cycle of length at most seven created by the identification. Then it corresponds to a v_1-v_3 path $P = v_1x_1 \ldots x_kv_3$ in G, where $k \leq 6$. If neither v_2 nor v_4 is on the path P, then there must be a separating cycle of length at most nine, a contradiction to (5). Hence v_2 is on the path P, without loss of generality, assume $v_2 = x_1$. If $k \leq 5$, the cycle $x_1x_2 \ldots x_kv_3v_2$ is a cycle of length at most six, it has a common edge v_1v_2 with the cycle $v_1v_2v_3v_4v_1$ in G, which is impossible. So, $|C^*| = 7$ and $k = 6$. Since the cycle $x_1x_2 \ldots x_6v_3v_2$ is not adjacent to any 3-cycle in G for $G \in \mathcal{G}_{(A)}$, and hence it is not adjacent to any 3-cycle in G^*. Similarly, edges v_3v_4, v_4v_1, v_1v_2 is not adjacent to any 3-cycle in G^*, because they all lie in a 4-cycle of G and $G \in \mathcal{G}_{(A)}$. Therefore, C^* is not adjacent to any 3-cycle in G^* and hence $G^* \in \mathcal{G}_{(A)}$.

By the minimality of G, the precoloring of C_0 can be extend to a proper 3-coloring of G^*, which just corresponds to a proper 3-coloring of G, a contradiction. □

(13) The graph G has no inner 6-faces.

Proof. Let f be an inner 6-face, with boundary a 6-cycle $\partial(f) = v_1v_2v_3v_4v_5v_6v_1$. Obviously, by (6), there exists at least one vertex of $\{v_1, v_2, v_3, v_4, v_5, v_6\}$, say v_1, is not on C_0. By (8), either v_2 or v_6 is not on C_0, we assume that v_2 is not on C_0. Let G^* be the graph obtained from G by identifying the vertices v_1 with v_5 and v_2 with v_4. Because neither v_1 nor v_2 is on C_0, the cycle C_0 is also the outer cycle of the graph G^*.

We show that the outer cycle C_0 has no chord in G^*. Otherwise, we assume that there exists a chord in G^*, without loss of generality, we assume that v_4 is on C_0 and v_2 is not on C_0 but it has a neighbor v on C_0. Because the edge v_2v_3 is contained in the 6-cycle $v_1v_2v_3v_4v_5v_6v_1$, it can not be contained in any 3-cycle, so v_3 is in $\text{int}(C_0)$. By the nonadjacency condition, the cycle $C_0[v, v_4]v_3v_2v$ has length at least six, and $|C_0[v, v_4]| \geq 3$. The cycle $C_0[v_4, v]v_2v_3v_4$ is a separating cycle, and $|C_0[v_4, v]| \geq 9$. Hence $|C_0| \geq 3 + 9 = 12$, a contradiction.

We can also show that the identification does not make short cycles of G^* adjacent, that is to say, $G^* \in \mathcal{G}_{(A)}$.

Now G^* is a graph having the nonadjacency property $\mathcal{A}$ and G^* is a smaller graph than G, then the precoloring of C_0 can be extend to a proper 3-coloring of G^*, which obviously corresponds to a proper 3-coloring of G, a contradiction. □

(14) Suppose that f is a 5-face with boundary $\partial(f) = v_1v_2v_3v_4v_5v_1$, and both v_1 and v_3 are on the outer cycle C_0, then there exists a 7-cycle C' such that $E(C') \cap \{v_3v_4, v_4v_5, v_5v_1\} \neq \emptyset$.

Proof. As 5-cycles are chordless, vertices v_1 and v_3 are not adjacent in G. By (8), the vertex v_2 is on the cycle C_0. By (6), edges v_1v_2 and v_2v_3 are consecutive on C_0. Hence the vertices v_4, v_5 are in $\text{int}(C_0)$. By (8), the vertex v_4 has a neighbor distinct from v_5, in $\text{int}(C_0)$. So the cycle $C_0[v_3, v_1]v_5v_4v_3$ is a separating cycle, and then $|C_0[v_3, v_1]| \geq 9$. So, the cycle C_0 is a 11-cycle.

Let $C = v_1v_2v_3x_4 \dots x_kv_1$ be a cycle of length at most nine distinct from the 5-cycle $v_1v_2v_3v_4v_5v_1$. Clearly, $C \neq C_0$, there exists at least one vertex in $\text{ext}(C)$. If one of $\{v_4, v_5\}$ is not on C, the cycle C is a separating cycle of length at most nine, a contradiction. Therefore, both v_4 and v_5 are on the cycle C. The two vertices v_4 and v_5 divide the path $C[v_3, v_1]$ into three segments, at least one of the three segments is a path with length more than one. By the nonadjacency condition, this path is of length at least six. Hence $|C| = 2 + |C[v_3, v_1]| \geq 2 + 2 + 6 = 10$, a contradiction. Then every cycle of G containing edge v_1v_2 and v_2v_3 must be of length at least ten except the cycle $v_1v_2v_3v_4v_5v_1$. That is, every path linking v_1 and v_3 in $G - \{v_2\}$ is of length at least eight except the path $v_3v_4v_5v_1$.

Case 1: The vertices v_1 and v_3 receives different colors in the precoloring.

Delete the vertex v_2 and its incident edges v_1v_2 and v_2v_3, add a new edge v_1v_3, we obtain a new graph G^*. Obviously, the precoloring of C_0 corresponds to a proper 3-coloring of the outer cycle of G^*.

We next show that $G^* \in \mathcal{G}_{(A)}$. If there exist two cycles C_1 and C_2 violates the nonadjacency condition in G^*, then one of $\{C_1, C_2\}$, say C_1, must contain the edge v_1v_3. Since the path linking v_1 and v_3 in $G - \{v_2\}$ is of length at least eight except the path $v_3v_4v_5v_1$, then $C_1 = v_1v_3v_4v_5v_1$ and the cycle C_2 does not contain the edge v_1v_3, consequently, the cycle C_2 is a cycle of $G - \{v_2\}$. By the violated condition in G^*, we have $E(C_2) \cap \{v_3v_4, v_4v_5, v_5v_1\} \neq \emptyset$

and $|C_2| \leq 6$. Therefore, the 5-cycle $v_1v_2v_3v_4v_5v_1$ has a common edge with the cycle C_2 in G, a contradiction. Hence, G^* is a graph having the nonadjacency property $\mathcal{A}$.

By the minimality of G, the precoloring of the outer cycle of G^* can be extend to a proper 3-coloring of G^*, which corresponds to a proper 3-coloring of G, a contradiction.

Case 2: The vertices v_1 and v_3 receive the same color in the precoloring.

Delete the vertex v_2 together with its incident edges v_1v_2 and v_2v_3, and identify the vertices v_1 with v_3, then we obtain a new graph G^*. Obviously, the precoloring of C_0 corresponds to a proper 3-coloring of the outer cycle of G^*.

If $G^* \in \mathcal{G}_{(A)}$, the proper 3-coloring of the outer cycle of G^* can be extended to a proper 3-coloring of G^*, which corresponds a proper 3-coloring of G. So, $G^* \notin \mathcal{G}_{(A)}$. In other words, the identification do violate the nonadjacency condition. Then there exist two cycles C_1 and C_2 of length at most seven which are adjacent in G^*. If both C_1 and C_2 are cycles of G, this contradicts the nonadjacency condition in G. Thus, there exists a path of length at most seven linking v_1 and v_3 in the graph $G - \{v_2\}$. It must be the path $v_1v_5v_4v_3$, because in the graph $G - \{v_2\}$, the path linking v_1 and v_3 is of length at least eight except the path $v_3v_4v_5v_1$; and the other cycle C' is a cycle of G with length seven. Moreover, $E(C') \cap \{v_3v_4, v_4v_5, v_5v_1\} \neq \emptyset$.

□

(15) Suppose that f is a 5-face with boundary $\partial(f) = v_1v_2v_3v_4v_5v_1$, and either v_1 or v_3 is not on the outer cycle C_0, then there exists a 7-cycle C^* such that $E(C^*) \cap \{v_3v_4, v_4v_5, v_5v_1\} \neq \emptyset$.

Proof. Without loss of generality, assume that v_1 is not on the outer cycle C_0. Let G^* be the graph obtained from G by identifying the vertices v_1 with v_3. Clearly, the identification dose not damage the outer cycle C_0.

First, we show that the identification dose not create a chord of C_0. Otherwise, the vertex v_1 has a neighbor v on the cycle C_0 and the vertex v_3 is on the outer cycle C_0. By (8) and the nonadjacency condition, the vertices v_2 and v_5 are in $\mathrm{int}(C_0)$. Since the cycle $C_0[v, v_3]v_4v_5v_1v$ is a separating cycle of G, then $|C_0[v, v_3]| \geq 8$. Similarly, the cycle $C_0[v_3, v]v_1v_5v_4v_3$ is a separating cycle of G, and $|C_0[v_3, v]| \geq 8$. Hence $|C_0| \geq 16$, a contradiction. So the identification does not create a chord of C_0, the precoloring of C_0 is also a proper 3-coloring of the outer cycle of G^*.

If G^* is a graph having the nonadjacency property $\mathcal{A}$, then the precoloring of C_0 can be extend to a proper 3-coloring of G^*, and the coloring corresponds to a proper coloring of G, a contradiction. Then there exists two cycles that violate the nonadjacency condition in G^*. Clearly, one of them must the triangle $v^*v_4v_5$, where v^* is the vertex obtained by the identifying v_1 with v_3, and the other cycle C^* must be a cycle of G. By the nonadjacency in G, the cycle C^* is a cycle of G with length seven, and it has a common edge with the

triangle $v^*v_4v_5$. That is, there exists a cycle C^* of length seven in the graph G such that $E(C^*) \cap \{v_3v_4, v_4v_5, v_5v_1\} = \emptyset$. □

Finally, we use the discharging method to get a contradiction and finish the proof of the lemma.

The Euler formula: for the plane graph G, $|V(G)| - |E(G)| + |F(G)| = 2$, can be written as following:

$$\sum_{v \in V(G)} (\deg_G(v) - 4) + \sum_{f \in F(G)} (\deg(f) - 4) = -8.$$

Initially, set the charge of every vertex $v \in V(G)$ by $w(v) = \deg_G(v) - 4$, and the charge of every face $f \neq f_0$ by $w(f) = \deg(f) - 4$ and $w(f_0) = \deg(f_0) + 4$. Obviously, the total sum of the initial charges is zero, i.e.,

$$\sum_{x \in V(G) \cup F(G)} w(x) = 0.$$

The discharging rule:

(R1) Each inner 3-face receives charge $1/3$ from each incident vertex.

(R2) If $\deg_G(v) = 5$, the vertex v sends charge $1/15$ to each incident 7^+-face.

(R3) If $\deg_G(v) \geq 6$, the vertex v sends charge $1/3$ to each incident face.

(R4) For all the inner vertices v:

- (a) If $\deg_G(v) = 3$ and v is incident with a 3-face, then v receives charge $2/3$ from each incident non-triangular face;
- (b) If $\deg_G(v) = 3$ and v is incident with a 5-face, then v receives charge $1/5$ from the 5-face and receives charge $2/5$ from each non-5-face;
- (c) If $\deg_G(v) = 3$ and v is not incident with 3- or 5-faces, then v receives charge $1/3$ from each incident face;
- (d) If $\deg_G(v) = 4$ and v is incident with exactly one 3-face, but not incident to any 5-face, then v receives charge $1/3$ from the incident face non-adjacent to the 3-face;
- (e) If $\deg_G(v) = 4$ and v is incident with only one 3-face, v is incident to a 5-face, then v receives charge $1/15$ from the incident face adjacent to the 3-face, receives charge $1/5$ from the 5-face;
- (f) If $\deg_G(v) = 4$ and v is incident with two 3-face, then v receives charge $1/3$ from each incident non-triangular face;

(g) If $\deg_G(v) = 4$ and v is not incident with 3-face, but it is incident with 5-faces, then v receives charge $1/5$ from each 5-face and sends charge $1/15$ to each incident 5^+-face.

(R5) For all the outer vertices v:

(a) if $\deg(v) = 2$ and v is incident with an inner 5-face, then v receives charge $3/5$ from the inner 5-face and receives charge $7/5$from the outer face;

(b) if $\deg(v) = 2$ and v is incident with an inner face having degree at least seven, then v receives charge $2/3$ from the inner face and receives charge $4/3$ from the outer face;

(c) if $\deg(v) = 3$, then the vertex v receives charge $4/3$ from the outer face;

(d) if $\deg(v) = 4$, then the vertex v receives charge $2/3$ from the outer face;

(16) After the discharging process, all the vertices have nonnegative final charges.

Remark 1. By the discharging rule, if a face f sends charge $2/5$ to its incident vertex v_3, then the vertex v_3 has degree three, and it is incident with a 5-face, see Fig. 5. If $\deg_G(v_2) \geq 4$, then the face f sends to the vertex v_2 at most $1/15$. If $\deg_G(v_2) = 3$, the face f sends charge $2/5$ to the vertex v_2, and then it follows from (14, 15) and the fact $G \in \mathcal{G}_{(A)}$ that either v_4 or v_1 is not bad; note that in this case three non-bad vertices are consecutively on the face boundary.

(17) For all the face f, the final charge of f is nonnegative. Moreover, the final charge of the outer face is positive.

Proof. Consider the outer face f_0. Assume that there are l outer vertices receiving charge $7/5$ from the outer face. Obviously, $l \leq 5$. Therefore, the final charge of f_0 is at least $\deg(f_0) + 4 - \frac{7}{5}l - \frac{4}{3}(\deg(f_0) - l) = -\frac{1}{3}\deg(f_0) + 4 - \frac{1}{15}l > 0$.

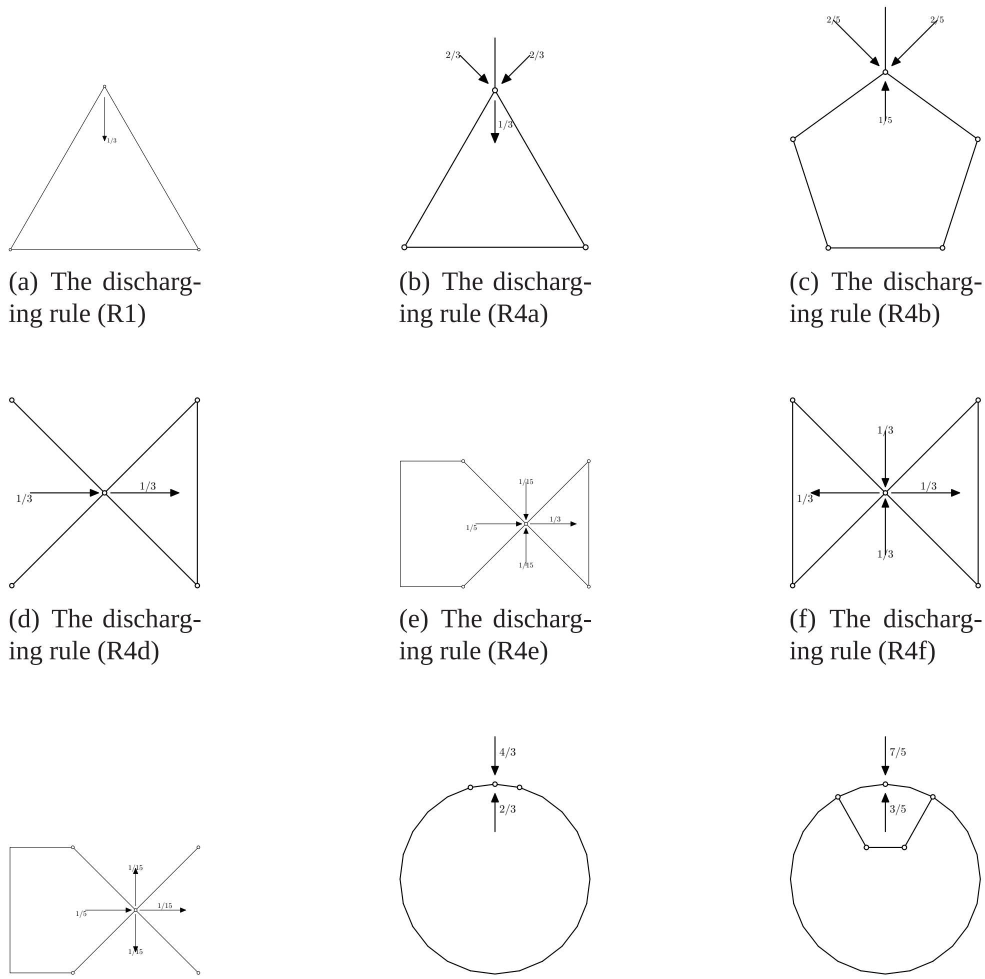
If f is an inner 3-face, then the final charge of f is at least $3 - 4 + 3 \times \frac{1}{3} = 0$.

If f is an inner 5-face, and the boundary of f contains a 2-vertex, then the face sends nothing to two incident vertices, see Fig. 3(i), the final charge of f is at least $5-4-\frac{3}{5}-2\times\frac{1}{5} = 0$.

If f is an inner 5-face, and the boundary of f contains no 2-vertices, then the final charge of f is at least $5 - 4 - 5 \times \frac{1}{5} = 0$.

Let f be an inner 7-face. By (8) and the hypothesis that 3-cycles are not adjacent to 7-cycles, the boundary of f contains at least two vertices in $\mathrm{int}(C_0)$, and the face f send to each such vertex by at most $2/5$.

If f is an inner 7-face which is not incident with a 2-vertex, then the final charge of f is at least $7 - 4 - 7 \times \frac{2}{5} = \frac{1}{5} > 0$; if f is an inner 7-face which is incident with a 2-vertex,



(a) The discharging rule (R1)

(b) The discharging rule (R4a)

(c) The discharging rule (R4b)

(d) The discharging rule (R4d)

(e) The discharging rule (R4e)

(f) The discharging rule (R4f)

(g) The discharging rule (R4g)

(h) The discharging rule (R5a)

(i) The discharging rule (R5b)

Fig. 4: The discharging process.

then f sends nothing to at least two vertices on C_0, and hence the final charge of f is at least $7-4-3\times\frac{2}{3}-2\times\frac{2}{5}=\frac{1}{5}>0$.

Let f be an inner face with degree at least eight. If the face f is incident with a 2-vertex, it sends nothing to at least two vertices on C_0. Thus the final charge of f is at least $\deg(f)-4-\frac{2}{3}(\deg(f)-2)\geq 0$. Now we assume that the boundary of an arbitrary inner face with degree at least eight contains no 2-vertices. Hence if a face sends a $2/3$ to its incident vertex, the vertex must be an inner bad vertex.

Let f be an inner face with degree at least ten. It contains at most $\deg(f)-2$ bad vertices

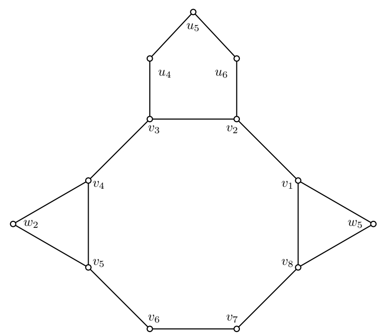


Fig. 5: A big face is incident with a 5-face.

by (11). If the face f does not send $2/5$ to its incident vertex, then the final charge of f is at least $\deg(f) - 4 - \frac{2}{3}(\deg(f) - 2) - 2 \times \frac{1}{3} \geq 0$. If the face f send $2/5$ to its incident vertex, then there are at most $\deg(v) - 3$ bad vertices on the boundary by (9), the final charge of f is at least $\deg(f) - 4 - \frac{2}{3}(\deg(v) - 3) - 3 \times \frac{2}{5} \geq \frac{2}{15} > 0$.

Then we only have to consider the inner 8-faces and 9-faces.

Let f be an inner 9-face. By (11), the boundary of face f contains at most seven bad vertices. If the boundary of f contains seven bad vertices, then the other two vertices separate the seven bad vertices as $4 + 3$ by (11), and the four bad vertices does not form a tetrad by (10). The local structure must be as in Fig. 6, and then the final charge of f is at least $9 - 4 - 7 \times \frac{2}{3} - \frac{1}{3} = 0$.

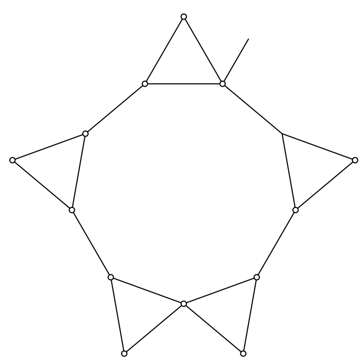


Fig. 6:

If the boundary of f contains six bad vertices and f does not send $2/5$ to its incident vertices, then the final charge of f is at least $9 - 4 - 6 \times \frac{2}{3} - 3 \times \frac{1}{3} = 0$. If the boundary of f contains six bad vertices and f sends charge $2/5$ to a vertex, then the final charge of f is at least $9 - 4 - 6 \times \frac{2}{3} - \frac{2}{5} - \frac{1}{3} - \frac{1}{15} > 0$ by Remark 1 and (9). If the boundary of f contains at most five bad vertices, then the final charge of f is at least $9 - 4 - 5 \times \frac{2}{3} - 4 \times \frac{2}{5} > 0$.

Finally, we dealt with the 8-face f. By (11), the boundary of f contains at most six bad vertices.

If the boundary of f contains six bad vertices, then the other two vertices separate the six bad vertices as $4+2$ or $3+3$ by (11). Note that these two non-bad vertices are not consecutively, so the face doesn't sends $2/5$ to the two non-bad vertices.

(i) The six bad vertices are separated by the other two vertices into two segments, where one contains four bad vertices and the other contains two bad vertices.

The four bad vertices v_1, v_2, v_3, v_4 does not form a tetrad, then v_1v_8 and v_4v_5 are in triangles. If v_6v_7 is in a triangular face, then f will send nothing to the vertices v_5 and v_8, its final charge is at least $8-4-6\times\frac{2}{3}=0$. If v_5v_6, v_7v_8 are respectively in a triangular face, the face f is a two-ear face, see Fig. 7, a contradiction(the detail is leaving for the reader, you can also see [6]).

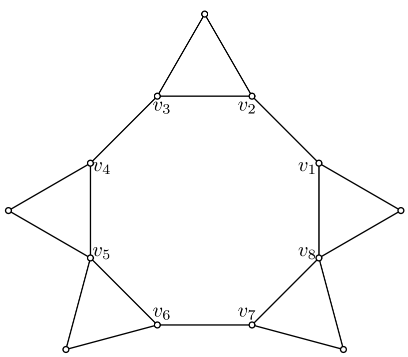


Fig. 7: Two ear face

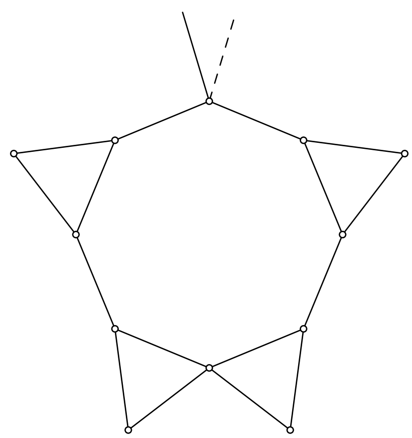


Fig. 8: One ear face

(ii) The six bad vertices are separated by the other two vertices into two segments, each of which contains three bad vertices.

It is not too hard to see that the local structure is a one-ear face, see Fig. 8, a contradiction.

Suppose now the boundary of f contains five bad vertices. First, assume that f sends $2/5$ to its incident vertex v_3, see Fig. 5. There exists a vertex v_2 on $\partial(f)$, such that v_3v_2 is in a 5-cycle. If $\deg(v_2)\geq 5$, then the face f receives from the vertex v_2 at least $1/15$. Hence the final charge of f is at least $8-4-5\times 2/3-2/5+1/15-1/3=0$. If $\deg(v_2)=3$, then the three non-bad vertices are consecutively on the boundary by Remark 1, and the five bad vertices lie consecutively on the boundary, a contradiction to (11).

Then we assume that $\deg_G(v_2) = 4$. By (11), vertices v_1, v_4 are bad and the edge v_4v_5 is in a triangle. Then, it follows from (14, 15) that the edge v_1v_8 is in a triangle. By (9), the non-bad vertex is one of $\{v_6, v_7\}$. But the edge v_6v_7 is in a triangle, so the non-bad vertex is of degree at least four. By the discharging rule, the face f sends nothing to the non-bad vertex. Hence the final charge of the face is at least $8 - 4 - 5 \times 2/3 - 2/5 > 0$.

Then assume f does not send charge $2/5$ to its incident vertices. If f sends nothing to at least one vertex, the final charge of the face is at least $8 - 4 - 5 \times 2/3 - 2 \times 1/3 > 0$. If not, by the discharging rules, the bad vertices are paired linked by the edges in the triangle, a contradiction to the fact that 5 is odd.

In the end, we may assume the boundary of f contains four bad vertices, because the final charge of f is no less than $8-4-3\times 2/3-5\times 2/5 = 0$ if f contains at most three bad vertices. If f does not send charge $2/5$ to its incident vertices, then the final charge of f is at least $8 - 4 - 4 \times 2/3 - 4 \times 1/3 = 0$. Then we assume that f does sends charge $2/5$ to its incident vertex v_3, and the edge v_2v_3 is in a 5-face. If there exists a non-bad vertex receiving from f at most $1/15$, then the final charge of f is at least $8-4-4\times 2/3-1/15-3\times 2/5 > 0$. Then we assume that every non-bad vertex receives charge from f greater than $1/15$, then $\deg_G(v_2) = 3$, or v_2 receives charge no more than $1/15$ from f. By (9), one of $\{v_4, v_5\}$ is a non-bad vertex, and one of $\{v_8, v_1\}$ is a non-bad vertex. By (14, 15), without loss of generality, we assume that the v_3v_4 is in a 7-face, then v_4 is not bad and v_5 is bad. Moreover, v_5v_6 is in a triangular. Face f sends charge great than $1/15$ to v_4, by the discharging rule, v_4 is of degree three, but this contradicts (9). $\square$

We complete the proof of the color extension lemma. $\square$

Proof of Theorem 2.1. Suppose the theorem is not correct. Let G be a minimum counterexample. Then G is simple, 2-connected, and with girth less than six. Hence, it must has a cycle C_0 of length less than six. If C_0 is an outer cycle of G, a contradiction to the extension Lemma. If C_0 is a separating cycle, we can first color the cycle C_0, and thus extend the coloring to $\mathrm{int}(C_0)$ and $\mathrm{ext}(C_0)$, and yields a proper 3-coloring of G, a contradiction. If C_0 is a inner facial cycle, then we can redraw the graph G, such that C_0 is the outer cycle, then apply the extension Lemma, a contradiction. $\square$

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